Handout 2 Lin Alge Notes

Definitions

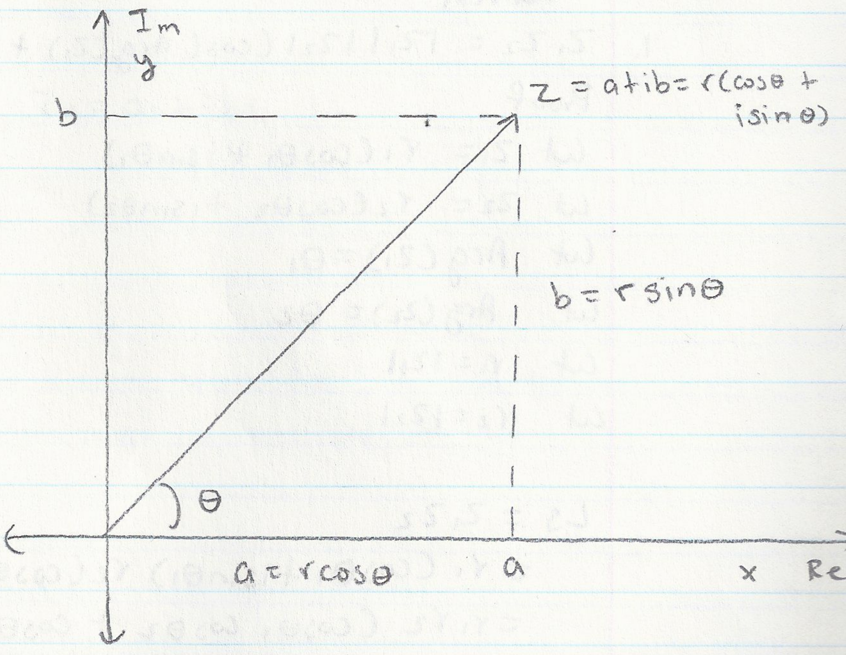
1. The set of Complex Numbers, denoted by $\mathbb{C}$, is the set of all numbers of the form $x+iy$ where $x, y \in \mathbb{R}$. x is the real part and b is the imaginary part.
$i = \sqrt{-1}$

2. The modulus or magnitude of an imaginary number, z, is denoted by $|z|$ and is the distance to the origin. Let $z = a+ib$
$|z| = \sqrt{a^2+b^2}$

3. The polar form of z is $z = r(\cos\theta + i\sin\theta)$, where r is the modulus of z.

4. The angle θ is called the argument of z. If $-\pi \le \theta \le \pi$, then θ is the principal argument of z. We denote the argument of z by $\mathrm{Arg}(z)$.
If $z = a+ib$, then $\theta = \frac{b}{a}$ and $\mathrm{Arg}(z) = \arctan(\theta)$.

When graphing complex numbers, the y-axis is the imaginary axis (Im) and the x-axis is the real axis (Re).



5. If $z = a+ib$, then is conjugate, denoted by $\bar{z}$, is $a-ib$.

Complex Addition

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then $z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$

Complex Multiplication

$z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= x_1 x_2 + i x_1 y_2 + i x_2 y_1 + i^2 y_1 y_2$$
$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

Scalar Complex Multiplication

Let r be a scalar

$rz = rx + iry$

Theorem

1. $z_1 z_2 = |z_1||z_2|(\cos(\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)) + i\sin(\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)))$

Proof

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

Let $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Let $\mathrm{Arg}(z_1) = \theta_1$

Let $\mathrm{Arg}(z_2) = \theta_2$

Let $r_1 = |z_1|$

Let $r_2 = |z_2|$

$$LS = z_1 z_2$$
$$= r_1(\cos\theta_1 + i\sin\theta_1) r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2 (\cos\theta_1 \cos\theta_2 + \cos\theta_1 i\sin\theta_2 + i\sin\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2)$$
$$= r_1 r_2 (\cos(\theta_1 + \theta_2) + i(\sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1)))$$
$$= r_1 r_2 (\cos(\theta_1 + \theta_2) + i(\sin(\theta_1 + \theta_2)))$$
$$= |z_1||z_2|(\cos(\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)) + i\sin(\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2))$$
$$= RS$$

2. $\dfrac{z_1}{z_2} = \dfrac{|z_1|}{|z_2|}\left(\cos(\text{Arg}(z_1) - \text{Arg}(z_2)) + i\sin(\text{Arg}(z_1) - \text{Arg}(z_2))\right)$

Proof

$\dfrac{z_1}{z_2}$

$= \dfrac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \dfrac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$

$= \dfrac{r_1}{r_2} \dfrac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$

$= \dfrac{|z_1|}{|z_2|} \dfrac{\cos\theta_1\cos\theta_2 - i\sin\theta_2\cos\theta_1 + i\sin\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}{\cos^2\theta_2 - i\sin\theta_2\cos\theta_2 + i\sin\theta_2\cos\theta_2 + \sin^2\theta_2}$

$= \dfrac{|z_1|}{|z_2|}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right)$

$= \dfrac{|z_1|}{|z_2|}\left(\cos(\text{Arg}(z_1) - \text{Arg}(z_2)) + i\sin(\text{Arg}(z_1) - \text{Arg}(z_2))\right)$

$=$ RS

Let $z = a + ib$, $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$

3. $\bar{\bar{z}} = z$

$\bar{z} = (a - ib)$

$\bar{\bar{z}} = a + ib$

$= z$

$=$ RS

4. $|z|^2 = z\bar{z}$

RS $= (a+ib)(a-ib)$

$= a^2 - aib + aib - i^2b^2$

$= a^2 + b^2$

$= |z|^2$

$=$ LS

3. $\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$

$LS = \overline{(a_1+ib_1)+(a_2+ib_2)}$

$= \overline{(a_1+a_2)+i(b_1+b_2)}$

$= (a_1+a_2)-i(b_1+b_2)$

$= (a_1-ib_1)+(a_2-ib_2)$

$= RS$

4. $z^{-1} = \dfrac{\overline{z}}{|z|^2}$

$LS = z^{-1}$

$= \dfrac{1}{z}\dfrac{(\overline{z})}{(\overline{z})}$

$= \dfrac{\overline{z}}{|z|^2}$

$= RS$

5. $\overline{z_1/z_2} = \overline{z_1} / \overline{z_2}$

$LS = \overline{z_1/z_2}$

$= \dfrac{\overline{z_1}}{\overline{z_2}}\ \dfrac{z_2}{z_2}$

$= \dfrac{\overline{z_1} \cdot z_2}{|z_2|^2}$

$= \overline{z_1}\left(\dfrac{1}{z_2}\right)$

$= \dfrac{\overline{z_1}}{z_2}$

$= RS$

6. $Re(z) = \frac{z+\bar{z}}{2}$

$Re(z) = a$

$RS = \frac{(a+ib)+(a-ib)}{2}$

$= \frac{2a}{2}$

$= a$

$LS = RS$

7. $Re(z) \leq |z|$

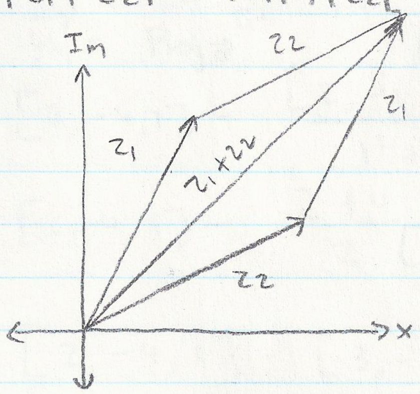
$LS = Re(z)$

$= a$

$|z| = \sqrt{a^2+b^2}$

$a \leq \sqrt{a^2+b^2}$

8. $|z_1+z_2| \leq |z_1|+|z_2|$



Since we know that in a triangle, no side length is greater than the sum of the other 2 side lengths,

$|z_1+z_2| \leq |z_1|+|z_2|$

Hints

1. To find the shortest distance from the point $(0,2,-3)$ to the line that goes thru $(1,-1,-2)$ and $(2,-2,-2)$.

Let $\vec{u} = [2,-2,-2] - [1,1,-2]$
$= [1,-3,0]$

Let $\vec{a} = [2,-2,-2] - [0,2,-3]$
$= [2,-4,1]$

$\vec{u} - \text{Proj}_{\vec{a}} \vec{u}$

$$= [1,-3,0] - \frac{[2,-4,1]\cdot[1,-3,0]}{2^2+(-4)^2+1^2}[2,-4,1]$$

$$= [1,-3,0] - \frac{2+12}{4+16+1}[2,-4,1]$$

$$= [1,-3,0] - \frac{14}{21}[2,-4,1]$$

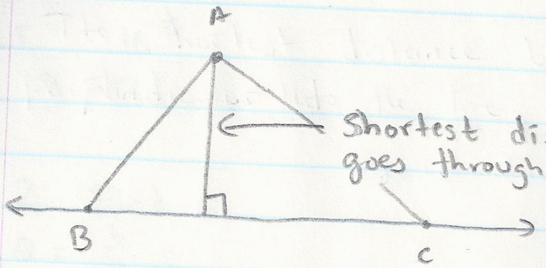
$$= [1,-3,0] - \frac{2}{3}[2,-4,1]$$

$$= [1,-3,0] - [\tfrac{4}{3}, \tfrac{-8}{3}, \tfrac{2}{3}]$$

$$= [\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{2}{3}]$$

Hint

1. How to find the shortest distance from the point (0, 2, -3) to the line that goes through the points (1, -1, -2) and (2, -2, -2).



Shortest distance is a line $\perp$ to the line that goes through B and C

Let $\vec{u} = [2,-2,-2] - [0,2,-3]$
$= [2,-4,1]$

Let $\vec{a} = [2,-2,-2] - [1,-1,-2]$
$= [1,-1,0]$

$\vec{u} - \text{Proj}_{\vec{a}}\vec{u}$

$= [2,-4,1] - \dfrac{[2,-4,1]\cdot[1,-1,0]}{1^2+(-1)^2}[1,-1,0]$

$= [2,-4,1] - \dfrac{2+4}{2}[1,-1,0]$

$= [2,-4,1] - [3,-3,0]$
$= [-1,-1,1]$

$\sqrt{(-1)^2+(-1)^2+(1)^2}$
$= \sqrt{3}$

2. Express $(1+i)^8$ in the form $a+bi$.

Step 1. Find r

$$r = \sqrt{a^2+b^2}$$
$$= \sqrt{1^2+1^2}$$
$$= \sqrt{2}$$

Step 2. Find θ

$$\theta = \frac{b}{a}$$
$$= 1$$

Step 3. Find Arg(z)

$$\text{Arg}(z) = \arctan(\theta)$$
$$= \arctan(1)$$
$$= \frac{\pi}{4}$$

Step 4. Write in polar form

$$z = r(\cos(\text{Arg}(z)) + i\sin(\text{Arg}(z)))$$
$$= \sqrt{2}\left(\cos\left(\tfrac{\pi}{4}\right) + i\sin\left(\tfrac{\pi}{4}\right)\right)$$

Step 5. Plug into the formula

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$
$$z^8 = (\sqrt{2})^8\left(\cos\left(8\left(\tfrac{\pi}{4}\right)\right) + i\sin\left(8\left(\tfrac{\pi}{4}\right)\right)\right)$$
$$= 16(1+0)$$
$$= 16$$

3. Find the four fourth roots of -16.

1. Find r

$$r = \sqrt{a^2+b^2}$$
$$= \sqrt{(-16)^2}$$
$$= 16$$
$$r^{\frac{1}{4}} = 16^{\frac{1}{4}}$$
$$= 2$$

2. Find θ

$$\theta = \frac{b}{a}$$
$$= \frac{0}{-16}$$
$$= 0$$

3. Find $\operatorname{Arg}(z)$

$$\operatorname{Arg}(z) = \arctan(\theta)$$
$$= \arctan(0)$$
$$= \pi$$

4. Sub into eqn

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right)$$
$$= 2\left(\cos\left(\frac{\pi}{4} + \frac{k\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{k\pi}{2}\right)\right)$$

$k = 0, 1 \ldots n-1$
$= 0, 1 \ldots 3$

5. Sub all the diff values for k

If $k=0$, $z = \ldots$
If $k=1$, $z = \ldots$
If $k=2$, $z = \ldots$
If $k=3$, $z = \ldots$